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Time-optimal control policies for cascaded production–inventory systems with control and state constraints†

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In this paper time-optimal control policies are derived for models of production–inventory systems consisting of a cascade of basic production–inventory systems with control and state constraints. The analytic solution is due to a decoupling of the complete system into its subsystems by a recursive definition of the cascaded system. It is shown that there is at least one bang-bang controlled subsystem. For the 'other' subsystems singular control policies are obtained. Introducing a pseudo-bang-bang control for these systems it is demonstrated that by strengthening the constraints there is a continuous transition from a singular to a bang-bang control.

1. Introduction

In a recent paper (Bradshaw and Erol 1980) sub-optimal control policies were derived for a class of linear time-invariant models of production–inventory systems consisting of a cascade of m basic production–inventory systems with bounded inputs. The control policies were obtained by the transformation of the continuous-time model into a discrete-time model and a subsequent application of a dead-beat control policy.

In this paper it is shown that the control policies can be derived by an analytic solution defining the cascaded system in a recursive manner. Often it is necessary to impose constraints on the state variables too. Concerning the production–inventory systems this case could arise if the inventories must be regarded with natural limits or if an overtension of the system must be avoided which leads to a restriction of the production rate. In this paper we consider the latter case.

The state constrained system is treated by the application of the theory developed by Jacobson *et al.* (1971), Hamilton (1972) and Maurer (1977). The theory is illustrated by the presentation of the state and control variables of two production–inventory systems in cascade.

2. Systems with linear control. Necessary conditions

Before treating the cascaded production–inventory system a short presentation of necessary conditions is given concerning state constrained systems. The state of the problem and the necessary conditions follow Maurer (1977). Thereby a modifcation is introduced by formulating the minimum principle for multiple input and multiple state constraint. Moreover, as will be clear in the sequel, only scalar input and scalar constraint will be necessary due to

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the decoupling of the system into its subsystems. So Maurer's theory holds for the following problem: determine a piecewise continuous control $u(t)$, $t\in[0, 1]$, which minimizes the functional

$$J(u) := F(x(1)) \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)) := f_1(x(t)) + f_2(x(t))u(t) \tag{2}$$

$$x(0) = x_0, \quad Q(x(1)) = 0 \tag{3}$$

$$u(t)\epsilon U := \{u \mid \zeta_j \leqq u_j \leqq \xi_j, \quad j = 1, \ldots, m\} \subset \mathbb{R}^m \tag{4}$$

$$S_i(x(t)) \leqq 0, \quad i = 1, \ldots, h \tag{5}$$

with the state vector $x(t)\in\mathbb{R}^n$, the control vector $u(t)\in\mathbb{R}^m$, the functions $F: \mathbb{R}^n\to\mathbb{R}$, $Q: \mathbb{R}^n\to\mathbb{R}^k$, $k\leqq n$, $f_1: \mathbb{R}^n\to\mathbb{R}^n$, $f_2: \mathbb{R}^n\to\mathbb{R}^{n\cdot m}$ and $S: \mathbb{R}^n\to\mathbb{R}^h$, $h\leqq m$, which are assumed to be sufficiently many times continuously differentiable.

Definition

The interval

$$I := [t_1, t_2] \subset [0, 1] \tag{6}$$

is called the boundary arc and t_1 and t_2 are called entry- and exit-time, respectively, of the trajectory $x(t)$ concerning S_i, if there exists a value $\epsilon > 0$ with $S_i(x(t)) \neq 0$ for $t_1 - \epsilon \leqq t < t_1$ and for $t_2 < t \leqq t_2 + \epsilon$ and $S_i(x(t)) = 0$ for $t\in I$. I is called interior arc if I is disjoint with any boundary arc.

Definition

For each component $S_i(x)$ of the constraint $S(x)$, $i = 1, \ldots, h$, let the order p_i be the smallest integer, such that the p_ith time derivative of $S(x(t))$ contains $u(t)$ explicitly.

For each optimal solution $x(t)$, $u(t)$ of the problem (1) to (5) there exists an adjoint vector $\lambda(t)\in\mathbb{R}^n$, a real vector $\sigma\in\mathbb{R}^k$ and a measurable function $\eta: [0, 1]\to\mathbb{R}^h$ with $\eta^T(t)S(x(t)) \equiv 0$, where T denotes the transpose, $t\in[0, 1]$, satisfying (7) to (11)

$$\dot{\lambda}^T = -\lambda^T f_{1x}(x) - \lambda^T f_{2x}(x)u - \eta^T S_x(x) \tag{7}$$

$$\lambda^T(1) = F_x(x(1)) + \sigma^T Q_x(x(1)) \tag{8}$$

This relation holds for problems fulfilling the Slater condition (see Girsanov 1972). The stated problem is assumed to satisfy this condition. Setting

$$\nu(t) := \eta(t^+) - \eta(t^-)$$

hence

$$\lambda^T(t^+) = \lambda^T(t^-) - \nu^T(t)S_x(x(t)), \quad \nu_i(t) \geqq 0 \tag{9}$$

Discontinuities ($\nu_i(t) > 0$) can only occur at entry- and at exit-points for S_i. With the hamiltonian

$$H(x(t), u(t), \lambda(t)) := \lambda^T(t)f_1(x(t)) + \lambda^T(t)f_2(x(t))u(t) + \eta^T(t)S(x(t)) \tag{10}$$

the minimum principle takes the form

$$H(x(t), u(t), \lambda(t)) = \min_{u \in U} H(x(t), u, \lambda(t)) \tag{11}$$

Autonomous systems with a free end time T may be transformed into a problem (1) to (5) by introducing an additional state variable and a subsequent substitution. At last we get

$$\dot{x}(t) = Tf_1(x(t)) + Tf_2(x(t))u(t), \quad t \in [0, 1] \tag{12}$$

and as further condition

$$H(x(1), u(1), \lambda(1)) = -1 \tag{13}$$

The switching function for the control u is

$$\phi^T(t) := \lambda^T(t) f_2(x(t)) \tag{14}$$

For each component $\phi_j^T(t) \neq 0$ the minimum principle yields to the bang-bang control

$$u_j(t) = \begin{cases} \zeta_j & \text{if } \phi_j(t) > 0 \\ \xi_j & \text{if } \phi_j(t) < 0 \end{cases} \tag{15}$$

3. Cascaded production–inventory systems

3.1. *State equations*

Any cascaded production–inventory system, consisting of m subsystems, can be represented by the block diagram shown in Fig. 1.

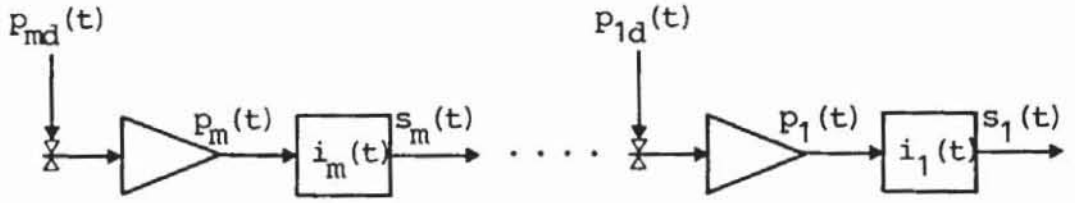

Figure 1.

The variables associated with the jth subsystem ($j = 1, \ldots, m$) have the following meanings : $p_{jd}(t)$ is the desired production rate ; $p_j(t)$ is the actual production rate ; $i_j(t)$ is the actual inventory level ; $s_j(t)$ is the shipping rate.

If it is assumed that the production process in each subsystem can be represented by the first order exponential delay with the time constant $1/\alpha_j$ ($j = 1, 2, \ldots, m$), then the governing equations for the subsystems are

$$\left.\begin{aligned} i_j(t) &= p_j(t) - s_j(t) && (16) \\ \dot{p}_j(t) &= \alpha_j(p_{jd}(t) - p_j(t)) && (17) \end{aligned}\right\} j = 1, \ldots, m$$

It is assumed that

$$s_{j+1}(t) = p_{jd}(t), \quad j = 1, \ldots, m \tag{18}$$

The aim is to find a control policy $p_{jd}(t)$ which transfers the system (16) to (18) from a given initial equilibrium state

$$\left.\begin{aligned} &i_m(0), i_{m-1}(0), \ldots, i_1(0) \\ &p_{md}(0)=p_m(0)=s_m(0)= \ldots =p_{1d}(0)=p_1(0)=s_1(0)=s_{10} \end{aligned}\right\} \tag{19}$$

to a given final equilibrium state

$$\left.\begin{aligned} &i_m(T), i_{m-1}(T), \ldots, i_1(T) \\ &p_{md}(T)=p_m(T)=s_m(T)= \ldots =p_{1d}(T)=p_1(T)=s_1(T)=s_{1T} \end{aligned}\right\} \tag{20}$$

in minimum time T, subject to the constraints

$$s_{1T}-\rho \leqq p_{jd}(t) \leqq s_{1T}+\rho, \quad j=1, \ldots, m, \quad \rho \in \mathbb{R}^+ \tag{21}$$

Throughout the paper we assume $\rho=1$. With

$$\left.\begin{aligned} u_j(t)&=p_{jd}(t)-p_{jd}(T) \\ x_{2j-1}(t)&=i_j(t)-i_j(T) \\ x_{2j}(t)&=p_j(t)-p_j(T) \end{aligned}\right\} \quad j=1, \ldots, m \tag{22}$$

and the transformation (12) the following state equations are obtained

$$\dot{x}(t)=TAx(t)+TBu(t) \tag{23}$$

where

$$x^{\mathrm{T}}=[x_1, \ldots, x_{2m}], \quad u^{\mathrm{T}}=[u_1, \ldots, u_m] \tag{24}$$

and

$$\left.\begin{aligned} A&=\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -\alpha_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & -\alpha_2 & \ldots & 0 & 0 \\ & & & & \vdots & & \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -\alpha_m \end{bmatrix} \\ B&=\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ \alpha_1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \alpha_2 & 0 & \ldots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & -1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \alpha_m \end{bmatrix} \end{aligned}\right\} \tag{25}$$

From (19), (20) and (22) we get the initial conditions

$$x_{2j-1}(0)=0, \quad x_{2j}(0)=s_{10}-s_{1T} \tag{26}$$

3.2. *Control policies for a submodel*

First let us consider an unbounded subproblem of the cascaded system (23) to (26).

(SP) We seek a control $u(t)\in\mathbb{R}$, $t\in[0, 1]$ with $|u(t)|\leqq 1$ and T minimal, such that the trajectory $x(t)\in\mathbb{R}^2$ is driven from $x^{\mathrm{T}}(0)=[0, a]$ to $x^{\mathrm{T}}(1)=[b, 0]$ according to the equation

$$\dot{x}(t)=T\begin{bmatrix}0 & 1\\ 0 & -\alpha\end{bmatrix}x(t)+T\begin{bmatrix}0\\ \alpha\end{bmatrix}u(t), \quad t\in[0, 1], \quad \alpha>0 \tag{27}$$

The hamiltonian is

$$H(x(t), u(t), \lambda(t))=(\lambda_1(t)-\alpha\lambda_2(t))x_2(t)+\alpha\lambda_2(t)u(t) \tag{28}$$

The adjoint equation becomes

$$\dot{\lambda}=-T\begin{bmatrix}0 & 0\\ 1 & -\alpha\end{bmatrix}\lambda. \tag{29}$$

It follows from (9) that λ is continuous and therefore one obtains

$$\lambda_1(t)=c \tag{30}$$

$$\lambda_2(t)=d\exp(T\alpha t)+c/\alpha \tag{31}$$

with some integration constants c and d.

The case $\lambda_2\equiv 0$ is excluded, since this would be a contradiction to (13). Furthermore λ_2 is a monotonous function with at most one zero. The switching function is $\phi(t)\geqslant\alpha\lambda_2(t)$ and therefore the system (27) turns out to be bang-bang controlled

$$u(t)=\begin{cases}1 & \text{if } \lambda_2(t)<0\\ -1 & \text{if } \lambda_2(t)>0\end{cases} \tag{32}$$

Transforming (27) into integral equations with respect to the initial conditions we obtain

$$x_1(t)=T\int_0^t x_2(\tau)\,d\tau \tag{33}$$

and

$$x_2(t)=\exp(-T\alpha t)T\alpha\int_0^t u(\tau)\exp(T\alpha\tau)\,d\tau+a\exp(-T\alpha t) \tag{34}$$

From (34) we get with the final conditions of (SP)

$$\int_0^1 u(\tau)\exp(T\alpha\tau)\,d\tau=-a/T\alpha \tag{35}$$

Equations (33) and (34) and the final conditions yield in accordance with the well known theorem of Fubini

$$b = T\left(\int_0^1 \exp(-T\alpha t) T\alpha \int_0^t u(\tau) \exp(T\alpha\tau)\, d\tau\, dt - a\,\frac{\exp(-T\alpha)-1}{T\alpha} \right)$$

$$= T\left(\int_0^1 \int_\tau^1 \exp(-T\alpha t) T\alpha u(\tau) \exp(T\alpha\tau)\, dt\, d\tau - a\,\frac{\exp(-T\alpha)-1}{T\alpha} \right)$$

$$= T\left(\int_0^1 u(\tau)\, d\tau - \exp(-T\alpha) \int_0^1 u(\tau) \exp(T\alpha\tau)\, d\tau - a\,\frac{\exp(-T\alpha)-1}{T\alpha} \right)$$

Together with (35) we now obtain

$$\int_0^1 u(\tau)\, d\tau = \frac{b}{T} - \frac{a}{T\alpha} \tag{36}$$

Because of (32) u is a bang-bang control, i.e. there exists a $t_s \in [0, 1]$ with

$$u(t) = \begin{cases} 1, & 0 \leqq t \leqq t_s \\ -1, & t_s < t \leqq 1 \end{cases} \tag{37}$$

or

$$u(t) = \begin{cases} -1, & 0 \leqq t \leqq t_s \\ 1, & t_s < t \leqq 1 \end{cases} \tag{37'}$$

Integrating (33) and (34) with (37) and (37') respectively the following two systems of equations turn out

$$\left.\begin{aligned} 2\exp(T\alpha t_s) - \exp(T\alpha) - 1 &= -a \\ 2t_s - 1 &= \frac{b}{T} - \frac{a}{T\alpha} \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} 2\exp(T\alpha t_s) - \exp(T\alpha) - 1 &= a \\ 2t_s - 1 &= \frac{a}{T\alpha} - \frac{b}{T} \end{aligned}\right\} \tag{38'}$$

From (38) and (38') respectively we take the solution with minimal time T as the optimal solution.

Now let us consider the bounded subproblem.

(SPB) This consists of the (SP) with the additional state constraint

$$S(x(t)) := x_2(t) - \beta \leqq 0, \quad t \in [0, 1] \tag{39}$$

The time derivative of $S(x(t))$ is

$$S^1(x(t)) = T[0, -\alpha]\, x(t) + T\alpha u = T\alpha x_2(t) + T\alpha u(t) \tag{40}$$

Thus $p = 1$ is the order of the constraint and if there exists any boundary arc I then $S^1(x(t)) = 0$ for $t \in I$ is valid. This yields to the boundary control :

$$u(t) = x_2(t) = \beta \tag{41}$$

Maurer (1977) has shown that the adjoint vector function is continuous at entry- and exit-points for constraints of order $p=1$. If the constraint is active, i.e. the solution of the (SP) violates (39), the switching function vanishes on the boundary arc : $\phi(t)=\lambda_2(t)\equiv 0$. Consequently the equation (7) takes the form

$$\dot{\lambda}=\begin{cases} -T\begin{bmatrix}0 & 0\\ 1 & -\alpha\end{bmatrix}\lambda & \text{at an interior arc}\\ \\ 0 & \text{at a boundary arc}\end{cases} \tag{42}$$

This yields to the monotonous function λ as for the (SP) and hence the structure of the optimal trajectory is ' interior arc '—' boundary arc '—' interior arc '. Thus the optimal control becomes

$$u(t)=\begin{cases} 1, & 0\leqq t<t_1\\ \beta, & t_1\leqq t<t_2\\ -1, & t_2\leqq t\leqq 1\end{cases} \tag{43}$$

In (34) we set $x(t_1)=\beta$ and get

$$t_1=\frac{\ln\dfrac{a-1}{\beta-1}}{T\alpha} \tag{44}$$

Applying (43) to (35) and (36) the following equations hold

$$t_1+\beta(t_2-t_1)+t_2-1=\frac{b}{T}-\frac{a}{T\alpha} \tag{45}$$

and

$$\exp(T\alpha t_1)-1+\beta(\exp(T\alpha t_2)-\exp(T\alpha t_1))+\exp(T\alpha t_2)-\exp(T\alpha)=-a \tag{46}$$

It is evident from (44), (45) and (46) that we are able to compute t_1, t_2 and T.

3.3. *Control policies for the cascade system*

Let us now proceed to the complete cascade problem. For the sake of simplicity the system (23) to (25) is driven from

$$\left.\begin{aligned} x^{\mathrm{T}}(0)&=[0, a, 0, \ldots, 0, a]\\ x^{\mathrm{T}}(1)&=[0, \ldots, 0]\end{aligned}\right\} \tag{47}$$

to

in minimal time T.

Analogously to (33) and (34), the trajectory is determined by

$$x_1(t)=T\int_0^t x_2(\tau)\,d\tau \tag{48}$$

$$x_{2j}(t)=\exp(-T\alpha_j t)T\alpha_j\int_0^t u_j(\tau)\exp(T\alpha_j\tau)\,d\tau+a\exp(-T\alpha_j t),\quad j=1, \ldots, m \tag{49}$$

$$x_{2j-1}(t)=T\int_0^t x_{2j}(\tau)\,d\tau-T\int_0^t u_{j-1}(\tau)\,d\tau,\quad j=2, \ldots, m \tag{50}$$

From (36) we know for $j=1$ with respect to (47)

$$\int_0^1 u_1(\tau)\,d\tau = \frac{-a}{T\alpha_1} \tag{51}$$

therefore we have

$$0 = x_3(1) = T\int_0^1 x_4(\tau)\,d\tau + a/\alpha_1 \tag{52}$$

Now we state the new system

$$\begin{bmatrix}\dot{\tilde{x}}_3\\ \dot{x}_4\end{bmatrix} = T\begin{bmatrix}0 & 1\\ 0 & -\alpha_2\end{bmatrix}\begin{bmatrix}\tilde{x}_3\\ x_4\end{bmatrix} + T\begin{bmatrix}0\\ \alpha_2\end{bmatrix}u_2 \tag{53}$$

with

$$\begin{bmatrix}\tilde{x}_3(0)\\ x_4(0)\end{bmatrix} = \begin{bmatrix}0\\ a\end{bmatrix},\quad \begin{bmatrix}\tilde{x}_3(1)\\ x_4(1)\end{bmatrix} = \begin{bmatrix}b_2\\ 0\end{bmatrix} \tag{54}$$

where $b_2 = -a/\alpha_1$.

The system (53) resembles (27) and thus it can be solved in the described way. With a solution (u_2, T), we obtain

$$x_3(t) = \tilde{x}_3(t) - T\int_0^t u_1(\tau)\,d\tau \tag{55}$$

Considering the final condition of x_5 we have

$$x_5(1) = T\int_0^1 x_6(\tau)\,d\tau - T\int_0^1 u_2(\tau)\,d\tau$$

With (36) the second term is

$$T\int_0^1 u_2(\tau)\,d\tau = b_2 - a/\alpha_2$$

Substituting b_2 we get for b_3 as final-condition of x_5

$$b_3 = -a(1/\alpha_1 + 1/\alpha_2)$$

Now we can proceed to the recursive definition of (23) to (25). Set

$$\left.\begin{aligned} b_1 &= 0\\ b_{j+1} &:= -a\sum_{i=1}^{j} 1/\alpha_i,\quad j=1,\ldots,m-1\end{aligned}\right\} \tag{56}$$

Introducing subsystems as follows

$$(\mathcal{S}_j),\quad j=1,\ldots,m$$

$$\begin{bmatrix}\dot{x}_{2j-1}\\ \dot{x}_{2j}\end{bmatrix} = T^j\begin{bmatrix}0 & 1\\ 0 & -\alpha_j\end{bmatrix}\begin{bmatrix}x_{2j-1}\\ x_{2j}\end{bmatrix} + T^j\begin{bmatrix}0\\ \alpha_j\end{bmatrix}u_j \tag{57}$$

with

$$\left.\begin{aligned} x_{2j-1}(0)=0, \quad x_{2j-1}(1)=b_j \\ x_{2j}(0)=a, \qquad x_{2j}(1)=0 \end{aligned}\right\} \tag{58}$$

the recursive definition is complete and therefore we are able to solve the entire system (23) to (25) analytically.

If the production rates x_{2j} are restricted the following constraint holds for each system

$$S_j(x)=x_{2j}-\beta_j \leqq 0 \tag{59}$$

These subsystems with bounded production rates are solvable by means described in § 2.

Let (t_s^j, T^j) in the case of unbounded subsystems and (t_1^j, t_2^j, T^j) for constrained subsystems be the optimal solution of $(\mathscr{S}_j)$. The adjoint equation related to the cascade problem is

$$\dot{\lambda}(t)=-A^{\mathrm{T}}\lambda(t) \tag{60}$$

and as switching function we have

$$\phi^{\mathrm{T}}(t)=(\alpha_1\lambda_2(t)-\lambda_3(t),\ \alpha_2\lambda_4(t)-\lambda_5(t),\ \ldots,\ \alpha_m\lambda_{2m}(t)) \tag{61}$$

Each component $\phi_j(t)$ is composed of a constant and an exponential function or zero, respectively. The assumption $\phi_j(t)\equiv 0$ for all $j=1, \ldots, m$ contradicts (13). Thus there exists a $j\in\{1, \ldots, m\}$, such that $u_j(t)$ is a bang-bang control which governs the subsystem $(\mathscr{S}_j)$. As a consequence the slowest subsystem $(\mathscr{S}_j)$ determines the time T for the entire cascade.

More formally : set $T := \max\{T^j \mid j=1, \ldots, m\}$. For all subsystems $(\mathscr{S}_j)$ with $T^j=T$ determine the bang-bang control u_j as described above (see (37), (43)). For all unbounded subsystems with $T^j<T$, i.e. which are governed by a singular control we introduce a ' pseudo-bang-bang control ' as follows

$$u_j(t) := \begin{cases} 1, & 0\leqq t<t_s^j \\ -1, & t_s^j\leqq t\leqq t_e^j \\ 0, & \text{otherwise} \end{cases} \tag{62}$$

This means that the ' faster ' subsystems are controlled as fast as possible by a bang-bang strategy in the first part of the control. In the second part the control is set to zero. Though there are many other possibilities to choose the singular controls it seems to be a reasonable strategy for a minimum time problem.

Thus for all subsystems with $T^j<T$ we seek the two switching points t_s^j and t_e^j. Therefore we have to solve the following equations

$$2\exp(T\alpha_j t_s^j)-\exp(T\alpha_j t_e^j)=-a \tag{63}$$

$$2Tt_s^j-Tt_e^j=b_j-a/\alpha_j \tag{64}$$

Concerning restricted subsystems we seek the entry-point t_1^j, the exit-point t_2^j and the end-point of the bang-bang control t_e^j with $0 \leqq t_1^j < t_2^j \leqq t_e^j \leqq 1$. With

$$u_j(t) := \begin{cases} 1, & 0 \leqq t < t_1 \\ \beta_j, & t_1 \leqq t \leqq t_2 \\ -1, & t_2 < t \leqq t_e \\ 0, & \text{otherwise} \end{cases} \tag{65}$$

the switching points are obtained by solving the following equations (see (44) (45), (46))

$$t_1^j = \frac{\ln \dfrac{1-a}{1-\beta_j}}{T\alpha_j} \tag{66}$$

$$t_1^j + \beta_j(t_2^j - t_1^j) + t_2^j - t_e^j = \frac{b_j}{T} - \frac{a}{T\alpha_j} \tag{67}$$

$$\exp(T\alpha_j t_1^j) - 1 + \beta_j(\exp(T\alpha_j t_2^j) - \exp(T\alpha_j t_1^j)) + \exp(T\alpha_j t_2^j) - \exp(T\alpha_j t_e^j) = -a \tag{68}$$

An important property of the chosen ' pseudo-bang-bang control ' is the fact that there is a continuous transition between singular and bang-bang controls. If the constraints of the singular controlled subsystems are strengthened, t_e^j approaches one. At the point $t_e^j = 1$ we have $T^j = T$ and the regarded subsystem becomes a bang-bang controlled one. If the constraint is strengthened further the subsystem remains bang-bang controlled and all other subsystems will be pseudo-bang-bang controlled.

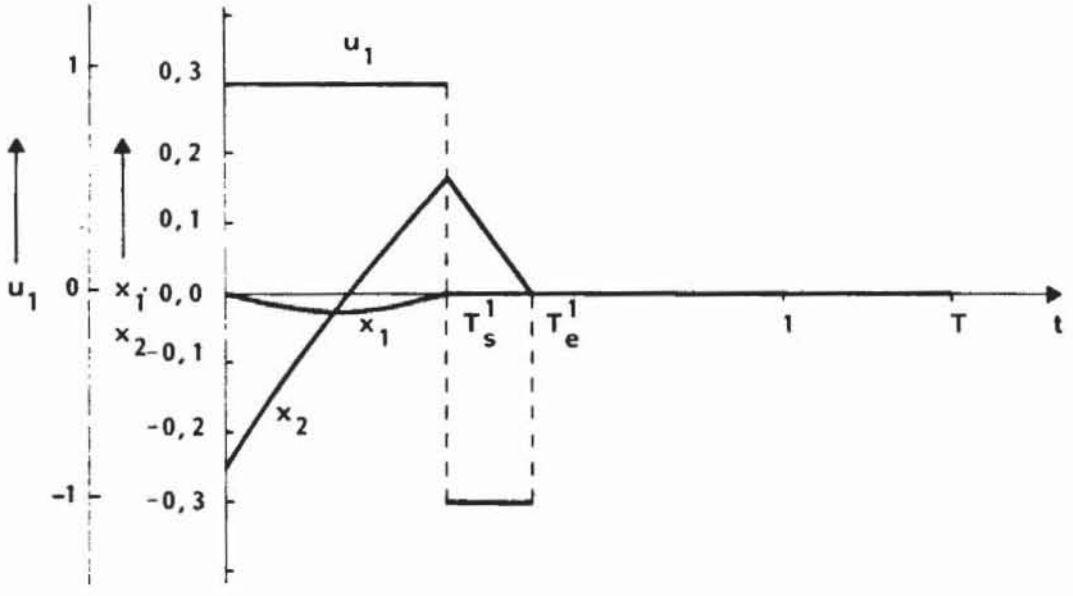

Figure 2.

Remark

If all production rates are unbounded the last subsystem $\mathscr{S}_m$ is bang-bang and the other systems are singular controlled.

3.4. *Numerical example*

As illustration of the theory presented in § 3.3 we consider a system consisting of two subsystems. For the ease of demonstration the parameters are assumed as follows : $\alpha_1 = \alpha_2 = 1$, $s_0 = -0{\cdot}25$, $s_T = 0{\cdot}0$. The time scale is let unnormed. Figures (2) and (3) show the history of the optimal control and state variables for the subsystems 1 and 2, respectively, in the unconstrained

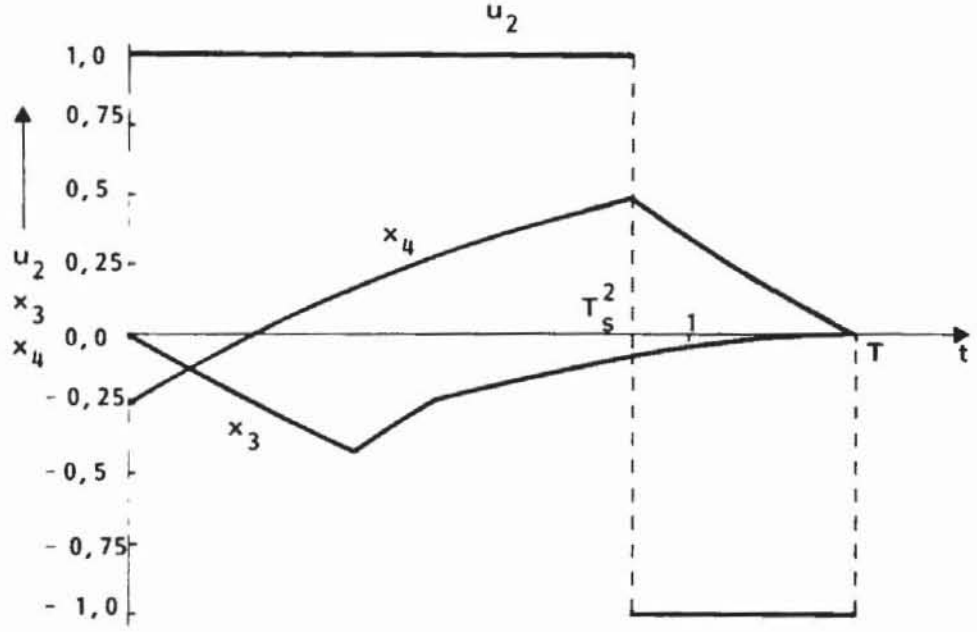

Figure 3.

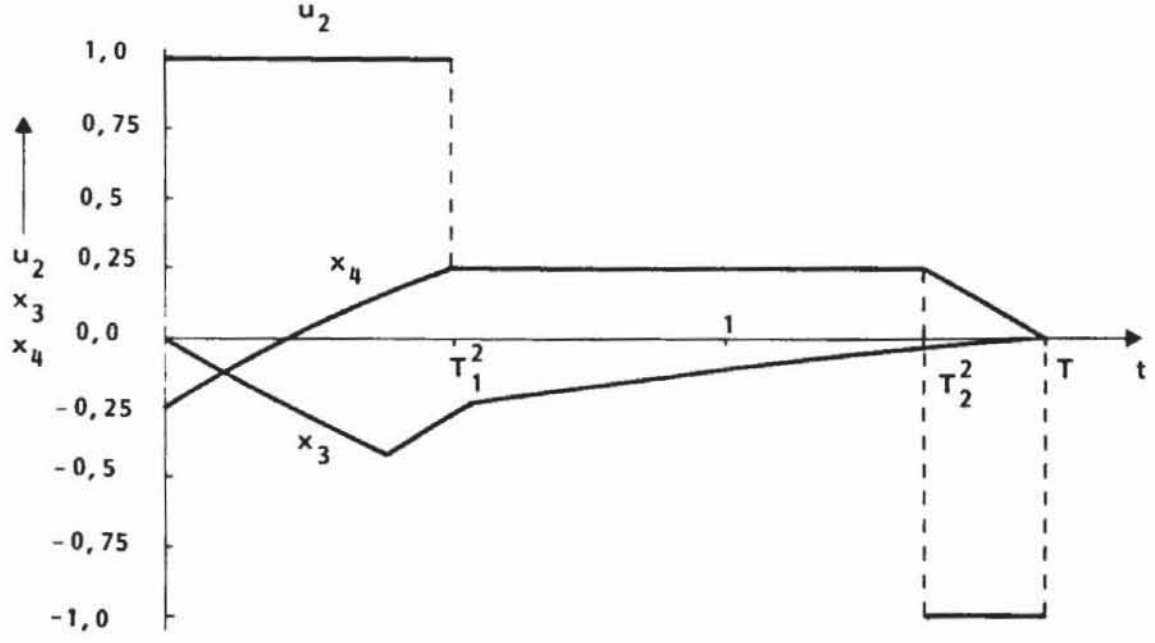

Figure 4.

case. The minimum time period is $T = 1{\cdot}2999$, the switching times for the first subsystem are $T_s^1 = 0{\cdot}4008$ and $T_c^1 = 0{\cdot}5516$. For the second subsystem the switching time is calculated as $T_s^2 = 0{\cdot}8999$. Figure (4) illustrates the constrained case. Assuming for the second production–inventory system a maximum production rate $\beta_2 = 0{\cdot}25$ which cannot be exceeded the state constraint takes the form $S_2(x(t)) = x_4(t) - 0{\cdot}25 \leqq 0$. The trajectories of the first subsystem remain exactly the same as in the unconstrained case, excepted the minimal time period which becomes $T = 1{\cdot}5832$. The variable $x_4(t)$ of the second subsystem lies on the boundary between the entry time $T_1^2 = 0{\cdot}5108$ and the exit time $T_2^2 = 1{\cdot}3601$.

In this paper time-optimal control policies have been derived for cascaded production–inventory systems with control and state constraints. It has been shown that the problem can be solved analytically due to a recursive definition of the cascaded system by means of the minimum principle for state constrained systems. It has been pointed out that at least one subsystem is bang-bang controlled whereas the others are singular controlled. By introducing a ' pseudo-bang-bang control ' the transition from bang-bang control to singular control and vice versa has been realized particularly for constrained problems.

The theory has been illustrated by the presentation of calculated results concerning two production–inventory subsystems in cascade for the unconstrained and the constrained case.

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